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PERIODIC SOLUTIONS FOR AN EQUATION GOVERNING DYNAMICS OF A RENEWABLE RESOURCE SUBJECTED TO ADDITIVE ALLEE EFFECTS IN A SEASONALLY VARYING ENVIRONMENT

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Abstract. In this article the minimum number of positive periodic solutions admitted by a non-autonomous scalar differential equation is estimated. This result is employed to find the minimum number of positive periodic solutions admitted by a model representing dynamics of a renewable resource that is subjected to additive Allee effects in a seasonally varying environment. Leggett-Williams multiple fixed point theorem is used to establish the existence of at least two positive periodic solutions for the considered dynamic equation. Two methods are obtained to establish the existence of periodic solutions. Key results are illustrated through numerical simulation.

Key Words. Renewable resource, additive Allee effect, Periodic solutions, positive solution

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1. Introduction.

The term Allee effect seems to have originated from the works of Allee [3, 4]. Allee effect refers to reduction in individual fitness at low population density that can lead to extinction [6, 11, 12]. It is strongly related to the extinction vulnerability of populations. According to [28], any ecological mechanism that can lead to a positive relationship between a component of individual fitness and either the number or density

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of conspecifics can be termed a mechanism of the Allee effect [19, 31] or depensation [10, 15, 21] or negative competition effect [37]. Several mechanisms generating Allee effects have been suggested in the literature[6, 12]. We are particulary interested in *additive Allee effect* [1, 2, 31, 32], among various Allee effects that exist, which refers to reduction of species due to an extra mortality rate influenced by factors such as satiation of a predator [7, 12, 14, 15, 34], anti predator behaviour like group defence against predator and inhibition [16, 29, 33, 41, 43] or the necessity of finding a mate for reproduction [7, 15, 22] etc. Two interesting derivations for the additive Allee effect can be found in [34] which are developed in the context of search for mate and impact of a satiating generalist predator.

Most ecosystems experience fluctuations in environmental factors which affect the birth rates, mortality rates, carrying capacities and other vital factors of the species in the ecosystems. In spite of the influence of such environmental fluctuations on species dynamics, the amount of analysis which has been carried out on autonomous growth models is much more than that which has been done on models in which the parameters are allowed to vary with time. In recent years an increasing amount of attention has been paid in both the biological and mathematical literature to the effects that such variations have on growth of species. It is evident that many of the variations with which the species must cope are regular and periodic. The quality and quantity of food and other vital resources, the occurrence of predation and competition, and the susceptibility or exposure to diseases or other hazards are but a few other examples of things which can affect growth and dynamics of species which can vary regularly [13, 27].

Thus periodicity and almost periodicity play an important role in the problems associated with real world applications. In trying to analyze the consequences of such periodic or almost periodic variations in the environment, it is reasonable as a first approximation to consider the involved parameters to be periodic of same period. Thus a natural approach might then be to study the effects of periodic variations in the appropriate parameters of the model equations which have been used to describe the growth dynamics in constant environments as in [8, 13, 30].

There has been considerable contributions in the recent years on the existence of periodic and almost periodic solutions of differential equations having periodic causal functions. In the recent works of Padhi and others [25, 26, 30] sufficient conditions have been obtained for the existence of multiple positive periodic solutions for certain differential equations using Leggett-Williams multiple fixed point theorem [20]. Existence of single and multiple periodic solutions of differential equations have been studied using fixed point theorems involving cone expansion and cone compression method, Upper-lower solution method and iterative techniques [9, 17, 35, 36, 47, 48]. Some relevant work on existence of at least one or at least two positive periodic solutions of scalar differential equations can be found [17, 42, 44, 45, 46]. Non existence of periodic solutions, existence of single and multiple periodic solutions for periodic functional differential equations have been established in [18, 38]. Existence of positive periodic solutions for systems of ordinary differential equations are presented in [23, 24]. Existence of positive periodic solutions for singular differential systems are established [39, 40].

In this article we are interested in investigating the existence of multiple positive periodic solutions of a first order differential equations representing growth of a renewable resource that is subjected to additive Allee effect in a seasonally varying environment. Our interest is to find an estimate on the number of positive periodic solutions admitted by the considered model. We have used Leggett-Williams multiple fixed point theorem [20] to establish the existence of at least two positive periodic solutions.

Section wise division of the article is as follows. In the next section we discuss about the model. In section 3 the existence of multiple positive periodic solutions for a general scalar differential equation is examined by assuming different conditions on the causal function. Results that guarantee existence of at least two positive periodic solutions for the considered resource dynamic equation are established in section 4. These results are obtained as an application of the general results presented in section 3. Section 5 illustrates the existence results through numerical simulation. This followed by discussion in section 6.

2. The Model. Let us consider the following differential equation representing the dynamics of a population subjected to additive Allee effects.

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k} - \frac{\nu}{1+\omega x}\right) \tag{1}$$

where the positive constants r and k stand for intrinsic growth rate and carrying capacity of the resource, ν and ω are constants that indicate the severity of Allee effect that has been modelled [1, 2]. Considering these parameters to be positive constants (which is a realistic assumption), it can be easily observed that (1) always admits the trivial solution as one of the equilibrium solutions and it admits at most two positive equilibrium solutions depending on the values of the remaining parameters. Equilibrium analysis and qualitative behaviour of solutions of (1) is presented in [34]. Assuming that the parameters r, k, ν, ω to be constants imply that the growth rate, carrying capacity, the extra mortality due to other factors remain the same throughout and are independent of the seasons. We introduce periodic variations in the growth dynamics by assuming that these parameters are periodic of same period [8, 13, 30].

In this article we are interested in studying the existence of positive periodic solutions of (1) under the assumption that the associated coefficients r, k, ν and ω are positive periodic functions of same period, P. Thus we consider the equation

$$\frac{dx}{d\tau} = r(\tau)x\left(1 - \frac{x}{k(\tau)} - \frac{\nu(\tau)}{1+\omega(\tau)x}\right). \tag{2}$$

The following lemma transforms the P periodic differential equation (2) into another equivalent simpler (involving only two periodic coefficients instead of three) T periodic differential equation where T could be different from P.
LEMMA 1. *The transformation* $t = G(\tau) = \int_0^\tau r(s)ds$ *transforms (2) to a* T *- periodic equation given by*

$$\frac{dy}{dt} = y\left(1 - \frac{y}{\kappa(t)} - \frac{\eta(t)}{1+m(t)y}\right) \tag{3}$$

with $y(t) = x(G^{-1}(t))$ *where* $\kappa(t) = k(G^{-1}(t))$, $\eta(t) = \nu(G^{-1}(t))$ and $m(t) = \omega(G^{-1}(t))$ *are positive periodic functions of period* $T = G(P)$. *Also, for each* T *- periodic solution* $y(t)$ *of (3),* $x(\tau) = x(G^{-1}(t))$ *defines a* P *- periodic solution of (2).*

In the light of lemma 1, we shall concentrate only on the existence of T periodic solutions for (3).

3. Periodic solutions for a general scalar differential equation.

In this section we shall study the existence of positive periodic solutions for a general scalar differential equation for which (3) becomes a special case. The results developed in this section will be applied to (3) to find conditions under which existence of two positive periodic solutions are guaranteed.

Let us consider the following general scalar differential equation

$$\frac{dy}{dt} = y + f(t,y) \tag{4}$$

where f, defined on $R \times R$, is a non positive valued continuous function and satisfies $f(t+T, y) = f(t,y)$. We have the following lemma which is easily verifiable.

LEMMA 2. *If $y(t)$ is a T - periodic solution of (4) then it also satisfies the integral equation*

$$y(t) = \int_{t}^{t+T} G(t,s) f(s, y(s)) ds \tag{5}$$

where $G(t,s)$ is the Green's function given by $G(t,s) = \frac{e^{-(s-t)}}{e^{-T}-1}$, $s \in [t, t+T]$. Note that the Green's function $G(t,s)$ satisfies

$$\frac{1}{e^{-T}-1} < G(t,s) < \frac{e^{-T}}{e^{-T}-1} < 0, \ s \in [t, t+T]. \tag{6}$$

The following fundamentals are needed to prove the results to follow. Let X be a Banach space and K be a cone in X. A mapping $\psi : K \to [0, \infty)$ is said to be concave nonnegative continuous functional [20] on K if it is continuous, nonnegative and satisfies

$$\psi(\eta x + (1-\eta) y) \geq \eta \psi(x) + (1-\eta)\psi(y), x, y \in K, \eta \in [0,1].$$

Let $a, b, c > 0$ be constants with K and X as defined above. Define

$$K_a = \{y \in K : ||y|| < a\}$$

and

$$K(\psi, b, c) = \{y \in K : \ ||y|| < c, \ \psi(y) > b\}.$$

THEOREM 1. *[Leggett-Williams multiple fixed point theorem] [20] : Suppose $E : \bar{K}_{c_3} \to K$ is completely continuous, and suppose there exists a concave non negative functional ψ with $\psi(y) \leq ||y||$, $y \in K$ and numbers c_1 and c_2, with $0 < c_1 < c_2 < c_3$ satisfying*
(i). $\{y \in K(\psi, c_2, c_3) : \psi(y) > c_2\} \neq \emptyset$ and $\psi(Ey) > c_2$ if $y \in \bar{K}(\psi, c_2, c_3)$
(ii). $||Ey|| < c_1$ if $y \in K_{c_1}$ and
(iii). $\psi(Ey) > \frac{c_2}{c_3}||Ey||$ for each $y \in K_{c_3}$ such that $||Ey|| > c_3$.
Then E has at least two fixed points in K_{c_3}.

It is well known that the set

$$Y = \{y \in C[0,T] : \ y(0) = y(T)\} \tag{7}$$

endowed with the norm

$$||y|| = \sup_{0 \leq t \leq T} |y(t)| \tag{8}$$

is a Banach space.

THEOREM 2. *Suppose that there exist two positive constants* $c_1 < c_2$ *such that*

(H_1) $\frac{e^{-T}}{e^{-T}-1}\int\limits_0^T f(s,y(s))ds \geq c_2$ for $c_2 \leq y(s) \leq \frac{c_2}{e^{-T}}$ for all $s \in [0,T]$,

(H_2) $\max\limits_{\|y\|\leq c_1}\left\{\frac{1}{e^{-T}-1}\int\limits_0^T f(s,y(s))ds\right\} < c_1.$

Then (4) admits at least two positive T-periodic solutions.

Proof. Let us consider the Banach space Y endowed with sup norm as defined in (7), (8). Define a cone K on Y as

$$K = \{y \in Y : y(t) \geq 0\}. \tag{9}$$

Let c_2 be the positive constant satisfying the conditions laid in the hypothesis. Define $c_3 = \frac{c_2}{e^{-T}}$. Define an operator $E : \bar{K}_{c_3} \to K$ as

$$(Ey)(t) = \int\limits_t^{t+T} G(t,s)f(s,y(s))ds$$

We shall apply *Leggett-Williams multiple fixed point theorem* to the above operator E to establish the existence of at least two positive periodic solutions for (4). It can be easily verified that E is well defined, completely continuous on $\bar{K}_{c_3}$ and $E(\bar{K}_{c_3}) \subset K$(Lemma 2).

Let us consider a concave non-negative continuous functional ψ on K defined as

$$\psi(y) = \min_{0\leq t\leq T} y(t). \tag{10}$$

Observe that $\psi_0 = \frac{1}{2}(c_2 + c_3)$ satisfies $c_2 < \psi_0 < c_3$ and belongs to $\{y \in K(\psi, c_2, c_3) : \psi(y) > c_2\}$.

Now, for $y \in \bar{K}(\psi, c_2, c_3)$ we have

$$\begin{aligned}
\psi(Ey) &= \min_{0\leq t\leq T}\int\limits_t^{t+T} G(t,s)f(s,y(s))ds \\
&> \frac{e^{-T}}{e^{-T}-1}\int\limits_0^T f(s,y(s))ds \quad \text{(from (6))} \\
&\geq c_2 \quad (\text{from}(\mathrm{H}_1)).
\end{aligned}$$

Therefore $\psi(Ey) > c_2$ for $y \in \bar{K}(\psi, c_2, c_3)$. Hence condition (i) of Theorem 1 is satisfied.

For any $y \in \bar{K}_{c_1}$ we have

$$\begin{aligned}
||Ey|| &= \sup_{0\le t\le T} \int_t^{t+T} G(t,s) f(s, y(s))ds \\
&< \frac{1}{e^{-T}-1} \int_t^{t+T} f(s, y(s))ds \quad \text{(from (6))} \\
&< \max_{||y||\le c_1} \left\{ \frac{1}{e^{-T}-1} \int_0^{T} f(s, y(s))ds \right\} \\
&< c_1 \quad \text{(from } (\mathrm{H}_2)).
\end{aligned}$$

Therefore $||Ey|| < c_1$ for $y \in \bar{K}_{c_1}$. Hence condition (ii) of Theorem 1 is satisfied. Below we shall establish the validity of condition (iii) of Theorem 1.

Let $y \in \bar{K}_{c_3}$ with $||Ey|| > c_3$. We have

$$\psi(Ey) = \min_{0\le t\le T} \int_t^{t+T} G(t,s) f(s, y(s))ds > \frac{e^{-T}}{e^{-T}-1} \int_t^{t+T} f(s, y(s))ds.$$

Therefore,

$$\frac{\psi(Ey)}{e^{-T}} > \frac{1}{e^{-T}-1} \int_t^{t+T} f(s, y(s))ds. \tag{11}$$

In view of (6) and (11) we obtain

$$\begin{aligned}
c_3 < ||Ey|| &= \sup_{0\le t\le T} \int_t^{t+T} G(t,s) f(s, y(s))ds \\
&< \frac{1}{e^{-T}-1} \int_t^{t+T} f(s, y(s))ds \\
&< \frac{\psi(Ey)}{e^{-T}} \quad \text{(from (11))}.
\end{aligned}$$

Therefore we have $||Ey|| < \frac{\psi(Ey)}{e^{-T}}$. Since $\frac{c_2}{c_3} = e^{-T}$ we have

$$\psi(Ey) > \frac{c_2}{c_3}||Ey||$$

for $0 < y(s) \leq c_3$ with $||Ey|| > c_3$. Hence condition (iii) of Theorem 1 is satisfied. Therefore by Theorem 1, (4) admits at least two positive T-periodic solutions. $\square$

COROLLARY 1. *Suppose that there exist two positive constants $c_1 < c_2$ such that*

$(\tilde{H}_1)$ $\quad \frac{e^{-T}}{e^{-T}-1} \int\limits_0^T f(s,y)ds = y \quad \text{at} \quad y = c_2$ *and*

$$\frac{e^{-T}}{e^{-T}-1} \int\limits_0^T f(s,y)ds > c_2 \quad \text{for} \quad c_2 < y \leq \frac{c_2}{e^{-T}}$$

$(\tilde{H}_2)$ $\quad \max\limits_{0<y\leq c_1} \left\{ \frac{1}{e^{-T}-1} \int\limits_0^T f(s,y)ds \right\} < c_1.$

Then (4) admits at least two positive T-periodic solutions.

THEOREM 3. *Assume that*

(H_1^*) $\quad \frac{e^{-T}}{e^{-T}-1} \int\limits_0^T f(s,y)ds$ *is strictly monotonically increasing such that*

$$\lim_{y\to\infty} \frac{e^{-T}}{e^{-T}-1} \frac{1}{y} \int\limits_0^T f(s,y)ds = \infty$$

(H_2^*) *there exists a constant $c > 0$ such that $\frac{1}{e^{-T}-1} \int\limits_0^T f(s,y)ds < c$ in a small left neighborhood of c.*

Then (4) admits at least two positive T-periodic solutions.

Proof. We shall show that the validity of conditions H_1^* and H_2^* imply validity of $\tilde{H}_1$ and $\tilde{H}_2$ of Corollary 1. Given that $\frac{e^{-T}}{e^{-T}-1} \int\limits_0^T f(s,y)ds$ strictly monotonically increasing with respect to y. Hence this monotonicity property holds for $\frac{1}{e^{-T}-1} \int\limits_0^T f(s,y)ds$ also. If (a,c) represents a left neighborhood in which H_2^* is valid then by choosing c_1 to be any element of (a,c) we have

$\max\limits_{0\le y\le c_1}\left\{\frac{1}{e^{-T}-1}\int\limits_0^T f(s,y)ds\right\} < c_1$. Thus $\tilde{H}_2$ is satisfied. From H_1^* and in view of H_2^* we have $\frac{e^{-T}}{e^{-T}-1}\int\limits_0^T f(s,y)ds < y$ over (a,c) and there exists a $c_2 > c$ such that $\frac{e^{-T}}{e^{-T}-1}\int\limits_0^T f(s,y)ds \ge c_2$ for all $y \ge c_2$. This implies validity of $\tilde{H}_1$. Hence, by Corollary 1, (4) admits at least two positive T-periodic solutions. ∎

The Theorem 2, Corollary 1 and Theorem 3 involve conditions on integral of the function $f(t,y)$. Below we present two existence theorems which are based on bounds of the function $f(t,y)$.

Theorem 4. *Suppose that there exists two positive constant $c_1 < c_2$ such that*

(H_3) $\min\limits_{0\le t\le T}\left\{\frac{e^{-T}}{e^{-T}-1}f(t,y(t))\right\} \ge \frac{c_2}{T}$ *when* $c_2 \le y(t) \le \frac{c_2}{e^{-T}}$ *for all* $t \in [0,T]$,

(H_4) $\max\limits_{\|y\|\le c_1}\left\{\max\limits_{0\le t\le T}\left\{\frac{1}{e^{-T}-1}f(t,y(t))\right\}\right\} < \frac{c_1}{T}$.

Then (4) admits at least two positive T-periodic solutions.

Corollary 2. *Suppose that there exists two positive constant $c_1 < c_2$ such that*

($\tilde{H}_3$) $\min\limits_{0\le t\le T}\left\{\frac{e^{-T}}{e^{-T}-1}f(t,y)\right\} = \frac{y}{T}$ *at* $y = c_2$ *and*

$$\min_{0\le t\le T}\left\{\frac{e^{-T}}{e^{-T}-1}f(t,y)\right\} > \frac{c_2}{T} \quad \textit{for} \quad c_2 < y \le \frac{c_2}{e^{-T}}$$

($\tilde{H}_4$) $\max\limits_{0\le y\le c_1}\left\{\max\limits_{0\le t\le T}\left\{\frac{1}{e^{-T}-1}f(t,y)\right\}\right\} < \frac{c_1}{T}$.

Then (4) admits at least two positive T-periodic solutions.

Theorem 5. *Assume that*

(H_3^*) $\min\limits_{0\le t\le T}\left\{\frac{e^{-T}}{e^{-T}-1}f(t,y)\right\}$ *is strictly monotonically increasing function such that* $\lim\limits_{y\to\infty}\min\limits_{0\le t\le T}\left\{\frac{e^{-T}}{e^{-T}-1}\frac{f(t,y)}{y}\right\} = \infty$

(H_4^*) *there exists a* $c > 0$ *such that* $\max\limits_{0\le t\le T}\left\{\frac{1}{e^{-T}-1}f(t,y)\right\} < \frac{c}{T}$ *in a small left neighborhood of* c.

Then (4) admits at least two positive T-periodic solutions.

Proofs of Theorem 4, Corollary 2 and Theorem 5 are parallel to that of Theorem 2, Corollary 1 and Theorem 3 respectively. Hence the proofs are omitted.

4. Application to renewable resource dynamics involving additive Allee effects. In this section, we shall apply the results developed in the previous section to find the existence of two positive periodic solutions of (3). Let us consider the following form of (3).

$$\frac{dy}{dt} = y - \frac{y^2}{\kappa(t)} - \frac{\eta(t)y}{1+m(t)y}. \tag{12}$$

In view of (4) we have

$$f(t,y) = -\frac{y^2}{\kappa(t)} - \frac{\eta(t)y}{1+m(t)y} \tag{13}$$

where the coefficients $\eta(t)$ and $m(t)$ are positive T-periodic continuous functions. We have

$$\begin{aligned}\lim_{y\to\infty} \frac{e^{-T}}{e^{-T}-1}\frac{1}{y}\int_0^T f(s,y)ds &= \lim_{y\to\infty} \frac{e^{-T}}{e^{-T}-1}\frac{1}{y}\int_0^T \Big\{-\frac{y^2}{\kappa(t)} - \frac{\eta(t)y}{1+m(t)y}\Big\}ds \\ &= \lim_{y\to\infty} \frac{e^{-T}}{1-e^{-T}}\int_0^T \Big\{\frac{y}{\kappa(t)} + \frac{\eta(t)}{1+m(t)y}\Big\}ds = \infty.\end{aligned}$$

Clearly H_1^* of Theorem 3 is satisfied by (3). We have the following theorem.
LEMMA 3. *If there exists a $l \geq 0$ such that*

$$\int_0^T \frac{\eta(s)}{1+m(s)l}ds < 1 - e^{-T} - l\int_0^T \frac{ds}{\kappa(s)} \tag{14}$$

then there is a $c > l$ that satisfies H_2^*. Further, (3) admits at least two positive T-periodic solutions.

Proof. Let us consider the equation

$$\frac{1}{e^{-T}-1}\int_0^T \Big(-\frac{y^2}{\kappa(s)} - \frac{\eta(s)y}{1+m(s)y}\Big)ds = y. \tag{15}$$

Simplifying (15) we obtain

$$\int_0^T \frac{\eta(s)}{1+m(s)y}ds = 1 - e^{-T} - y\int_0^T \frac{ds}{\kappa(s)}. \tag{16}$$

Let us denote

$$r(y) = \int_0^T \frac{\eta(s)}{1+m(s)y}ds,\ s(y) = 1-e^{-T}-y\int_0^T \frac{ds}{\kappa(s)} \text{ and } z(y) = \int_0^T \Big(\frac{y^2}{\kappa(s)} + \frac{\eta(s)y}{1+m(s)y}\Big)ds$$

defined on positive y axis. Observe that $z(y)$ is continuous and strictly increasing function, $r(y)$ is a convex function and it approaches zero as y approaches ∞. On the other hand $s(y)$ is a linear decreasing function intersecting the y axis at $(1-e^{-T})/\int_0^T \frac{ds}{\kappa(s)}$. Let us assume that there exists $l \geq 0$ such that $\int_0^T \frac{\eta(s)}{1+m(s)l}ds < 1-e^{-T}-l\int_0^T \frac{ds}{\kappa(s)}$. In view of the qualitative behaviour of the functions $r(y)$ and $s(y)$, there exists a $c \in \Big(l, (1-e^{-T})/\int_0^T \frac{ds}{\kappa(s)}\Big)$ such that $r(c) = s(c)$. Since $z(y) < z(c)$ for $y < c$ we have

$$\begin{aligned}\int_0^T \Big(\frac{y^2}{\kappa(s)} + \frac{\eta(s)y}{1+m(s)y}\Big)ds \ &< \ \int_0^T \Big(\frac{c^2}{\kappa(s)} + \frac{\eta(s)c}{1+m(s)c}\Big)ds \\ &= \ c(1-e^{-T}) \ \ (\text{since } r(c) = s(c)).\end{aligned}$$

Since $z(y) = \int_0^T \Big(\frac{y^2}{\kappa(s)} + \frac{\eta(s)y}{1+m(s)y}\Big)ds$ is strictly monotonically increasing continuous function we obtain

$$\int_0^T \Big(\frac{y^2}{\kappa(s)} + \frac{\eta(s)y}{1+m(s)y}\Big)ds < c(1-e^{-T}).$$

The above inequality can be rewritten as

$$\frac{1}{e^{-T}-1}\int_0^T \Big(-\frac{y^2}{\kappa(s)} - \frac{\eta(s)y}{1+m(s)y}\Big)ds < c. \tag{17}$$

Therefore H_2^* of Theorem 3 is satisfied. Hence, from Theorem 3, (3) admits at least two positive $T-$periodic solutions. ∎

Now we shall apply Theorem 5 to investigate the existence of positive periodic solutions for (3). Since the coefficient functions $m(t), \eta(t)$ and $\kappa(t)$ are assumed to be positive periodic functions, there exists positive constants a, b, d, f, g and h satisfying

$$a \leq \eta(t) \leq b, d \leq m(t) \leq f \quad \text{and} \quad g \leq \kappa(t) \leq h. \tag{18}$$

In view of (18) we have

$$\frac{y^2}{h} + \frac{ay}{1+yf} \leq \max_{0\leq t\leq T}\left\{\frac{y^2}{\kappa(t)} + \frac{\eta(t)y}{1+ym(t)}\right\} \leq \frac{y^2}{g} + \frac{by}{1+yd}. \tag{19}$$

Therefore we have

$$\begin{aligned}\lim_{y\to\infty}\min_{0\leq t\leq T}\frac{1}{y}\Big\{\frac{e^{-T}}{e^{-T}-1}f(t,y)\Big\} &= \lim_{y\to\infty}\min_{0\leq t\leq T}\frac{1}{y}\Big\{\frac{e^{-T}}{e^{-T}-1}\Big(-\frac{y^2}{\kappa(t)} - \frac{\eta(t)y}{1+m(t)y}\Big)\Big\}\\ &= \lim_{y\to\infty}\min_{0\leq t\leq T}\Big\{\frac{e^{-T}}{1-e^{-T}}\Big(\frac{y}{\kappa(t)} + \frac{\eta(t)}{1+m(t)y}\Big)\Big\} = \infty.\end{aligned}$$

Therefore H_3^* of Theorem 5 is satisfied. We have the following Lemma, proof of which can be constructed parallel to that of Lemma 3.
Lemma 4. *If there exists $l' \geq 0$ such that*

$$\frac{b}{1+l'd} < \frac{1-e^{-T}}{T} - \frac{l'}{g} \tag{20}$$

then there is a $c > l'$ that satisfies H_4^*. Further, (3) admits at least two positive $T-$periodic solutions.
Corollary 3. *The equation (3) admits at least two positive T-periodic solutions if the following quadratic equation in l'*

$$dTl'^2 - (gd(1-e^{-T}) - T)l' - g(1 - bT - e^{-T}) = 0 \tag{21}$$

admits at least one positive root.

5. Illustrative Example. In this section the key findings of the work are illustrated through numerical simulation. Let us consider the following model that describes dynamics of a fishery influenced by additive Allee effect in periodically fluctuating environment:

TABLE 1
Table presenting the description of various parameters along with their values and units pertaining to the problem (22)

Parameter	Description	value	Units
r	Intrinsic growth rate	0.25	1/year
K	Carrying capacity	3.5	million ton
σ_K	Amplitude of K fluctuation	1.5	million ton
η	parameter inducing additive Allee effect	2	-
σ_η	Amplitude of η fluctuation	0.9	-
m	parameter inducing additive Allee effect	6.5	1/million ton
σ_m	Amplitude of m fluctuation	0.5	1/million ton
Cycle	Environmental cycle	4	years

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{K - \sigma_k sin(\frac{2\pi\tau}{cycle})} - \frac{\eta - \sigma_\eta sin(\frac{2\pi\tau}{cycle})}{1 + (m - \sigma_m sin(\frac{2\pi\tau}{cycle}))x}\right). \tag{22}$$

Some of the forms for the coefficient functions are adopted from [5] and significance of various parameters considered in the problem (22) and their values are presented in the Table 1. Values for some of the parameters are adopted from the parameter set considered in [5].

Clearly the equation (22) is a $4 - year$ periodic differential equation. We shall employ the theory developed in the previous section to check if the equation (22) admits multiple periodic solutions. An application of Lemma 1 transforms equation (22) to the following equivalent $1-$ periodic differential equation:

$$\frac{dx}{dt} = x\left(1 - \frac{x}{K - \sigma_k sin(2\pi t)} - \frac{\eta - \sigma_\eta sin(2\pi t)}{1 + (m - \sigma_m sin(2\pi t))x}\right). \tag{23}$$

Now we search for multiple positive periodic solutions of equation (23). In view of Table 1 we have

$$\int_0^1 \frac{\eta - \sigma_\eta sin(2\pi t)}{1 + (m - \sigma_m sin(2\pi t))l} dt = \frac{2}{\sqrt{(1 + 6l)(1 + 7l)}}$$

and

$$\int_0^1 \frac{1}{K - \sigma_k sin(2\pi t)} dt = \frac{1}{\sqrt{10}}.$$

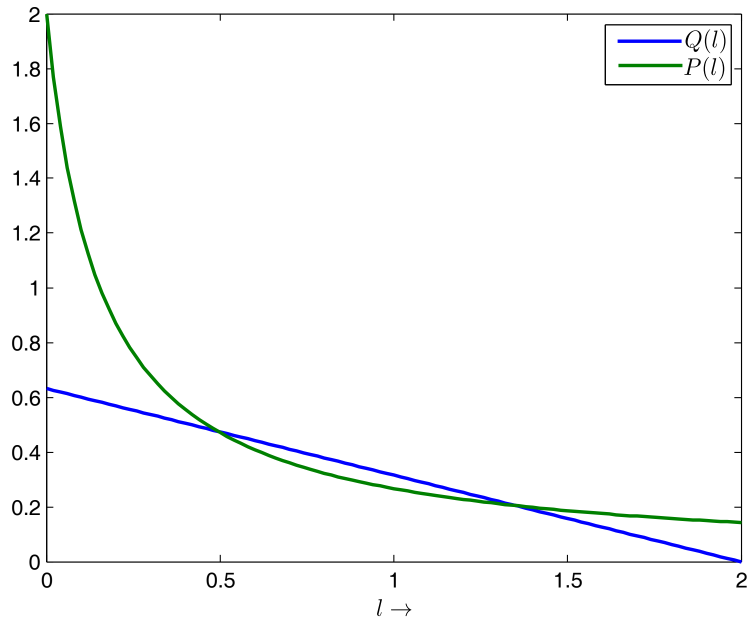

FIG. 1. *Figure presents the curves* $P(l) = \frac{2}{\sqrt{(1+6l)(1+7l)}}$ *and* $Q(l) = 1 - e^{-1} - \frac{l}{\sqrt{10}}$. *Observe that* $P(l) < Q(l)$ *in the vicinity of* 1 *which guarantees the existence of at least two positive periodic solutions for the equation (23) (Lemma 3)*

Now plotting the functions $P(l) = \frac{2}{\sqrt{(1+6l)(1+7l)}}$ and $Q(l) = 1 - e^{-1} - \frac{l}{\sqrt{10}}$ we observe that there exists a $\tilde{l} > 0$ such that $P(\tilde{l}) < Q(\tilde{l})$ (for example $\tilde{l}$ is 1, refer Fig 1). Hence by Lemma 3, the equation (23) admits at least two positive $1-$ periodic solutions and hence the equation (22) admits at least two $4 - year$ positive periodic solutions. It should be noted that the ease of integrating the functions $\frac{\eta - \sigma_\eta sin(2\pi t)}{1+(m-\sigma_m sin(2\pi t))l}$ and $\frac{1}{K-\sigma_k sin(2\pi t)}$ made the verification of Lemma 3 easier. In the absence of such facility for evaluation of integrals, one can use the Lemma 4 and make a conclusion regarding the existence of at least two positive periodic solutions. In any case Lemma 3 is superior to Lemma 4 as it can be observed that application of Lemma 4 does not guarantee the existence of two positive periodic solutions for (23) for the considered set of parameter values. However, Lemma 4 assures existence of two positive periodic solutions if the value of the parameter m is in the vicinity of 13 leaving the values of other parameters as in table 1. This can be verified from the existence of two positive roots for the equation (21).

Figure 2 presents the two $1-$ positive periodic solutions admitted by equation (23). It should be noted that the equation always admits the trivial

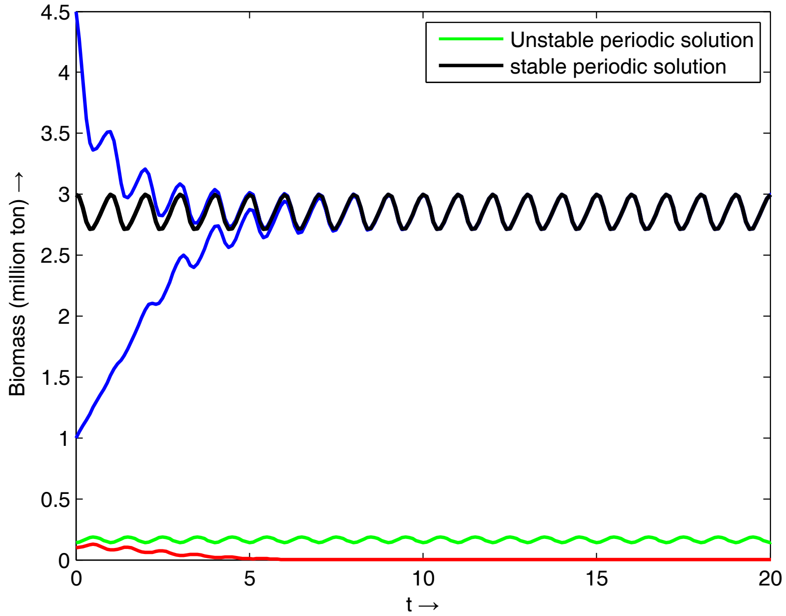

FIG. 2. *Figure presents the two positive periodic solutions admitted by the equation (23) for the parameter values taken from Table 1. The legend presented in the figure indicates the stability nature of each of the periodic solution. The other simulations shown blue and red colour illustrate the stability nature of the periodic solutions. From this illustration we observe that all the solutions initiating above the unstable periodic solution (blue colour trajectories)get attracted to the asymptotically stable periodic solution and those that initiate below the unstable periodic solution (red colour trajectory) approach the trivial solution asymptotically*

solution. The numerical simulations indicate that the equation (23) admits exactly two positive periodic solutions such that the smaller among them is unstable and the other is asymptotically stable. Further, all the positive solutions of (23), initiating below the unstable periodic solution approach the trivial solution asymptotically. On the other hand all the solutions initiating above the unstable periodic solution approach the locally asymptotically stable. In a subsequent paper we present theoretical results that establish existence of exactly two positive periodic solutions and their nature of stability.

6. Discussion and Conclusions. Allee effects occur when ever fitness of an individual in a small or sparse population decreases as the population size or density also declines [6, 11, 12, 31]. The additive Allee effect that is known to occur when a prey dynamics is influenced by predator satiation [7, 34], group defence in a prey species, inhibition in micro organisms [1, 2] or difficulty in searching for a mate [34]. In all the works that concerned the additive Allee effects, the involved parameters have been taken to be constants implying that the dynamics are time independent or the environment is constant in time. But in natural world a biological organism's physical environment is non constant in time. Often the environment is either periodic or almost periodic. Periodicity in the environment is incorporated into the dynamics of a species by assuming that the involved coefficients in the equation governing its dynamics to be periodic [8, 13, 30]. In this article we have considered dynamics of a renewable resource that is subjected to additive Allee effect in a periodically varying environment. We have observed that, under reasonable conditions on the coefficient functions, there are at least two positive periodic solutions for the considered model.

The existence of at least two positive periodic solutions is obtained by employing *Leggett-Williams Multiple fixed point theorem* to the considered model. The existence is established using two different conditions. The first type involves integral conditions while the other involves bounds of the periodic coefficients. The existence results are illustrated through numerical simulation in section 5 using a suitable example. The theoretical results developed in this article provide an upper bound on the number of positive periodic solutions admitted by the considered model. It would be more interesting to obtain results that decide the exact number of periodic solutions along with their stability nature. Work in this direction is in progress.

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REFERENCES

[1] Aguirre, P., Olivares, E.G., Saez,E., Two limit cycles in a Leslie-Gower predator-prey model with additive Allee effect. *Nonlinear Anal. Real World Appl.*, **10**(2009), 1401 – 1416.

[2] Aguirre, P., Olivares, E.G., Saez, E., Three limit cycles in a Leslie-Gower predator-prey model with additive Allee effect, *Siam J. Appl. Math.*, **69**(2009), # 5, 1244 – 1262.

[3] Allee, W.C. *Animal aggregations*, University of Chicago Press, Chicage, IL, 1931.

[4] Allee, W.C. *Cooperation among animals*, Henry Schuman, New York,1951.

[5] Anderson, L.G., Seijo, J.C. *Bio-economics of Fisheries Management*, Wiley-Blackwell publishing, Singapore, 2010.

[6] Berec,L., Angulo, E., Counchamp, F., Multiple Allee effects and population management, *Trends Ecol. Evol.*, **22**(2006), # 4, 185 – 191.

[7] Boukal, D.S., Berec, L., Single-species models of the Allee effect:extinction boundaries, sex ratios and mate encounters, *J. theor. Biol.*, **218**(2002), 375 – 394.

[8] César Castilho, Srinivasu, P.D.N., Bio-economics of a renewable resource in a seasonally varying environment, *Math. Biosc.*, **205**(2007), 1 – 18.

[9] Cheng, S.S., Zhang, G., Existence of positive periodic solutions for non-autonomous functional differential equations, *Electron. J. Differ. Equ.*, **59**(2001), 1 – 8.

[10] Clark, C.W., *Mathematical Bioeconomic: The optimal management of renewable resources (second edition)*, John Wiley and Sons, 1990.

[11] Courchamp,F., Brock, T.C.,Grenfell, B., Inverse dependence and the Alle effect, *Trends Ecol. Evol.*, **14**(1999), # 10, 405 – 410.

[12] Courchamp, F., Berec,L., Gascoigne,J., *Allee Effects in Ecology and Conservation*, Oxford University Press, Oxford, 2008.

[13] Cushing, J.M., Oscillatory population growth in periodic environments, *Theor. Popul. Biol.*, **30**(1986), 289 – 308.

[14] Dennis,B., Patil, G.P., The gamma distribution and weighted multimodel gamma distributions as models of population abundance, *Math. Biosci.*, **68**(1984), 187 – 212.

[15] Dennis, B., Allee effects: population growth, critical density, and the chance of extinction, *Nat. Resour. Model.*, **3**(1989),# 4, 481 – 538.

[16] Freedman, H.I., Wolkowicz, G.S.K., Predator-prey systems with group defence: The paradox of enrichment revisted, *Bull. Math. Biol.*, **8**(1986), 493 – 508.

[17] Jiang, D., Wei,J., Jhang, B., Positive periodic solutions of functional differential equations and population models, *Electron. J. Differ. Equ.*, **71**(2002), 1 – 13.

[18] Jin, Z.L., Wang, H., A note on positive periodic solutions of delayed differential equations, *Appl. Math. Let.*, *23*(2010), 581–584.

[19] Kent, A., Doncaster, C.P., Sluckin, T., Consequences for depredators of rescue and Allee effects on prey, *Ecol. Model.*, **162**(2003), 233 – 245.

[20] Leggett, R.W., Williamms, L.R., Multiple positive fixed points of non linear operators on ordered Banach spaces, *Indiana Univ. Math. J.*, **28**(1979), 673 – 688.

[21] Liermann, M., Hilborn, R., Depensation: evidence models and implications, *Fish*

and Fisheries, **2**(2001), 33 – 58.

[22] McCarthy, M.A., The Allee effect, finding mates and theoretical models, *Ecol. Model.*, **103**(1997), 99 – 102.

[23] O'Regan, D., Wang, H., Positive periodic solutions of systems of first order ordinary differential equations, *Result. Math.*, **48**(2005), 310–325.

[24] O'Regan, D.,Wang, H., Positive periodic solutions of systems of second order ordinary differential equations, *Positivity*, **10**(2006), 285–298.

[25] Padhi, S., Srivastava, S., Multiple periodic solutions for non linear first order functional differential equations with applications to population dynamics, *Appl. Math and comp.*, textbf203(2008), # 1, 1 – 6.

[26] Padhi, S., Srivastava, S., Dix, J.G., Existence of Three Nonnegative Periodic Solutions for Functional Differential Equations and Applications to Hematopoiesis, *Pan Amer. Math. J.*, **19**(2009),# 1, 27 – 36.

[27] Pianka,E.R., *Evolution ecology*, New York: Harper and Row, 1978.

[28] Rojas-Palma, A., Gonzalez-Olivares, E., Betsabe Gonzalez-yanez, Metastability in a gause type predator-prey models with sigmoid functional response and multiplicative Allee effect on prey, In R. Mondaini and R.Dilao (Eds) *Proccedings of the sixth Brazilian Symposium on Mathematical and computational Biology*, Epapers Servicos Editoriais Ltda, Rio de Janeiro, 2006.

[29] Ruan, S., Xiao, D., Global analysis in a predator-prey system with nonmonotonic functional response, *Siam J. Appl. Math.*, **61**(2001), 1445 – 1472.

[30] Padhi, S., Srinivasu, P.D.N., Kiran Kumar, G., Periodic Solutions for an equation governing dynamics of a Renewable Resource Subjected to Allee Effects, *Nonlinear Anal. Real World Appl.*, **11**(2010), 2610 – 2618.

[31] Stephens, P.A., Sutherland, W.J., Consequences of the Allee effect for behavior, ecology and conservation, *Trends Ecol. Evol.*, **14**(1999), # 10, 401 – 405.

[32] Stephens, P.A., Sutherland, W.J., *Vertebrate mating, Allee effects and conservation*, World scientific Publishing, London, 2000.

[33] Taylor, R.J., *Predation*, Chapman and Hall, 1984.

[34] Thiem, H.R., *Mathematics in Population Biology*, Princeton University press, Princeton and Oxford, 2003.

[35] Wan, A., Jiang, D., Existence of positive periodic solutions for functional differential equations, *Kvshu J.Math.*, **56**(2002), 193 – 202.

[36] Wan, A., Jiang, D., A new existence theory for positive periodic solutions to functional differential equations, *Comput. Math. Appl.*, **47**(2004), 1257 – 1262.

[37] Wang, G., Liang ,X.G., Wang, F.G., The competitive dynamics of populations subject to an Allee effect, *Ecol. Model.*, **124**(1999), 183 – 192.

[38] Wang, H., Positive periodic solutions of functional differential equations, *J. Diff. Equ.*, *202*(2004), 354–366.

[39] Wang, H., Positive periodic solutions of singular systems with a parameter, *J. Diff. Equ.*, **249**(2010), 2986–3002.

[40] Wang, H., Positive periodic solutions of singular systems of first order ordinary differential equations, *Appl. Math. Compu.*, **218**(2011), 1605–1610.

[41] Wolkowicz, G.S.W., Bifurcation analysis of a predator-prey system involving group defense, *Siam J. Appl. Math.*, **48**(1988), 592 – 606.

[42] Wu, X.M., Li, J.W., Zhou, H.Q., A necessary and sufficient condition for the existence of positive periodic solutions of a model of hematopoiesis, *Comput. Math. Appl.*, **54**(2007), 840 – 849.

[43] Xiao, D., Ruan, S., Bifurcations in a predator-prey system with group defense, *In-*

ternat. J. Bifur. Chaos., **11**(2001), 2123 – 2131.

[44] Yuji, L., Weigao, G., Positive periodic solutions of non linear differential equations, *Appl. Math. J. Chinese. Univ. Ser. B.*, **18**(2003), # 4, 373 – 382.

[45] Yuji, L., Further results on periodic boundary value problems for non linear first order impulsive functional differential equations, *J. Math. Anal. Appl.*, **327**(2007), 435 – 452.

[46] Zeng, Z., Bi, L., Fan, M., Existence of multiple positive periodic solutions for functional differential equations, *J. Math. Appl. Appl.*, **325**(2007), 1378 – 1389.

[47] Zhang, G., Cheng, S., Positive periodic solutions of non autonomous functional differential equations depending on a parameter, *Abstract and appl. Math.*, **7**(2003), 279 – 286.

[48] Zhang, W., Zhu, D., Bi, P., Existence of periodic solutions of a scalar differential equations via fixed point theorem, *Math. Comp. Modelling.*, **46**(2007), 718 – 729.